

On multiple positive ground state solutions for a mean curvature equation in Minkowski space

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Abstract

In this paper, we show how changes in the sign of nonlinearity leads to multiple radial ground state solutions of the mean curvature equation $\nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + \lambda f(u) = 0$ in \mathbb{R}^N for sufficiently large λ with $N \geq 2$.

Keywords. Ground state solution, multiplicity, radial solution, multiplicity

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1 Introduction

Hypersurfaces of prescribed mean curvature in Minkowski space are of interest in differential geometry and in general relativity. In this paper, we are concerned with the existence and multiplicity of such a kind of hypersurfaces which are graphs of the solution of the following problem

$$\begin{aligned} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + \lambda f(u) &= 0, & \text{in } \mathbb{R}^N, \\ u(x) &> 0, & \text{in } \mathbb{R}^N, \\ u(x) &\rightarrow 0, & \text{as } |x| \rightarrow \infty, \end{aligned} \quad (1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a local Lipschitz function with $f(0) = 0$, $\lambda > 0$ is a parameter and $N \geq 2$.

The differential operator we are considering has been deeply studied in the recent years, in nonlinear equations on bounded domains with various type of boundary conditions (see [1-5] and the references within) and in the whole \mathbb{R}^N (see [6,7]).

The radial solutions which only depend on $r = |x|$ of (1) satisfy the following ODE

$$\begin{aligned} \left(\frac{u'}{\sqrt{1-(u')^2}} \right)' + \frac{N-1}{r} \frac{u'}{\sqrt{1-u'^2}} + \lambda f(u) &= 0, \\ u(0) = \zeta, \quad u'(0) &= 0, \end{aligned} \quad (2)$$

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where $u \in C^2([0, +\infty])$ is now a function of $r = |x|$ alone, and ζ has to be determined in order to have

$$\lim_{r \rightarrow \infty} u(r) = 0. \quad (3)$$

The existence of the positive solution of (1) can be interpreted in this context as the existence of a ground state solution.

Recently, Azzollini [7] proves the existence of a ground state solutions of (1) with $\lambda = 1$ by the shooting method under the assumptions:

- (f1) $f(0) = 0$,
- (f2) $f : [0, +\infty) \rightarrow \mathbb{R}$ is locally Lipschitz,
- (f3) $\exists \alpha := \inf\{\xi > 0 \mid f(\xi) \geq 0\} > 0$,
- (f4) (if $N \geq 3$), $\lim_{s \rightarrow \alpha^+} \frac{f(s)}{s-\alpha} > 0$,
- (f5) $\exists \gamma > 0$ such that $F(\gamma) := \int_0^\gamma f(s)ds > 0$,
- (f6) $f(\xi) > 0$ in $(\alpha, \xi_0]$, where $\xi_0 := \inf\{\xi \in (0, \infty) \mid F(\xi) > 0\}$.

He proved the following

Theorem A. ([7, Theorem 0.1]) If

- $N \geq 3$ and f satisfies (f1)-(f6),
- $N = 2$ and f satisfies (f1)-(f3), (f5) and (f6),

then (1) has a radially decreasing solution with $\lambda = 1$.

The shooting argument has been used in the past to find ground state solutions to various types of equations. For examples, Berestycki, Lions and Peletier [8] study the existence of a ground state solution of the Laplace equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N \quad (4)$$

with $N \geq 2$. And the case $N = 1$, Berestycki and Lions [9] find the sufficient and necessary condition for the existence of the unique solution of the problem (4). Peletier and Serrin [10] are concerned with the existence of a ground state solution of the following prescribed mean curvature equation

$$\nabla \cdot \left[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right] - \lambda u + u^q = 0.$$

The shooting method consists in studying the profile of the solution of (2) as the initial value ζ varies into an interval. The main ideas is to exclude the cases in which for a finite $R > 0$ either u or u' vanishes.

On the other hand, Dávila del Pino and Guerra [11] find the problem

$$\Delta u - u + u^p + \lambda u^q = 0 \quad \text{in } \mathbb{R}^N$$

has at least three positive decaying radial solutions if $N = 3, 1 < q < 3, q < p < 5$ is taken sufficiently close to 5 and λ is fixed sufficiently large.

Naturally, what is really interesting is to find the conditions which permit to multiple ground state solutions of (1). Motivated above papers [6-11], this paper devotes to studying how changes in the sign of $f(s)$ leads to multiple positive radial solutions of (1).

We make the following assumptions:

(A1) $f : [0, +\infty) \rightarrow \mathbb{R}$ is locally Lipschitz with $f(0) = 0$;

(A2) there exists $2n$ real numbers $0 =: \beta_0 < \alpha_1 < \beta_1 < \dots < \alpha_n < \beta_n < \infty$ such that for $i \in \{1, \dots, n\}$,

$$f(s) < 0, \quad s \in (\beta_{i-1}, \alpha_i); \quad f(s) > 0, \quad s \in (\alpha_i, \beta_i);$$

(A2)' there exists $2n - 1$ real numbers $0 < \alpha_1 < \beta_1 < \dots < \alpha_n < \infty$ such that for $i \in \{1, \dots, n - 1\}$,

$$f(s) < 0, \quad s \in (\beta_{i-1}, \alpha_i); \quad f(s) > 0, \quad s \in (\alpha_i, \beta_i); \quad f(s) > 0, \quad s \in (\alpha_n, \infty);$$

(A3) let $F(u) := \int_0^u f(s)ds$. Then for each $i \in \{1, 2, \dots, n\}$, there exists $\xi_i \in (\alpha_i, \beta_i)$ such that $F(\xi_i) = 0$;

(A4) for each $i \in \{1, 2, \dots, n\}$, $F(\beta_{i-1}) < F(\beta_i)$;

(A5) (if $N \geq 3$) $\lim_{s \rightarrow \alpha_i^+} \frac{f(s)}{s - \alpha_i} > 0$, $i = 1, 2, \dots, n$ and $\lim_{s \rightarrow \beta_j^+} \frac{f(s)}{\beta_j - s} > 0$, $j = 1, 2, \dots, n - 1$.

In the sequel, we will suppose that f is extended in \mathbb{R} by setting $f(s) = 0$ if $s \leq 0$. Clearly, f is locally Lipschitz continuous on \mathbb{R} . The main result of the paper is the following:

Theorem 1. If

- $N \geq 3$ and f satisfies (A1)-(A5),
- $N = 2$ and f satisfies (A1)-(A4),

then (1) has n distinct radially decreasing solutions for $\lambda > 0$ is sufficiently large .

Remark 1. Note that Theorem 1 is Theorem A in the case $n = 1$ and $\lambda = 1$.

Remark 2. If we replace (A2) with (A2)', then the result of Theorem 1 is also true by a similar argument with obvious changes.

Remark 3. We exhibit some examples of functions f satisfying our assumptions: consider the function

$$f(s) = (s^2 - 19s + 18)(12s^2 - s^3 - 27s).$$

By a simple computation, we can get that f satisfies (A1)-(A5) with $\alpha_1 = 1, \alpha_2 = 9, \beta_1 = 3, \beta_2 = 18$. From Theorem 1, there exist numbers $\zeta_i \in (\xi_i, \beta_i)$, $i = 1, 2$ such that for sufficiently large λ , the problem (1) has two distinct positive, decaying radial solutions.

2 Proof of the main result

Since we are interested in the multiplicity of ground state solutions of (1), we aim to find n distinct numbers $\zeta_i \in (\xi_i, \beta_i)$, $i = 1, 2, \dots, n$ such that for $\lambda > 0$ is sufficiently large, the solution $u_i \in C^2(\mathbb{R}_+)$ of the IVP:

$$\begin{aligned} \left(\frac{u'}{\sqrt{1-(u')^2}} \right)' + \frac{N-1}{r} \frac{u'}{\sqrt{1-u'^2}} + \lambda f(u) &= 0, \\ u(0) = \zeta_i, \quad u'(0) &= 0 \end{aligned} \tag{5}_i$$

has the properties: $u_i(r) > 0$ for $r \in [0, \infty)$, $u'_i(r) < 0$ for $r \in (0, \infty)$ and

$$\lim_{r \rightarrow +\infty} u_i(r) = 0. \tag{6}$$

Observe that the solution of $(5)_i$ satisfies the equation

$$(r^{N-1} \phi'(u'))' = -r^{N-1} \lambda f(u), \tag{7}$$

where $\phi(s) := 1 - \sqrt{1-s^2}$ for $s \in [-1, 1]$. It is easy to verify that $\phi' : (-1, 1) \rightarrow \mathbb{R}$ is an increasing diffeomorphism. Set $\delta > 0$ and denote by $C := C([0, \infty), \mathbb{R})$ and by $C_\delta := C([0, \delta], \mathbb{R})$. Define the following operators

$$S : C \rightarrow C, \quad Su(r) := \begin{cases} -\frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t) dt, & \text{if } r > 0, \\ 0, & \text{if } r = 0, \end{cases}$$

and $K : C \rightarrow C$, $K(u)(r) = \int_0^r u(t) dt$.

For every $\zeta_i \in \mathbb{R}$, define the translation operator $T_{\zeta_i} : C \rightarrow C$ such that $T_{\zeta_i}(u) = \zeta_i + u$. Moreover, consider the Nemytskii operators associated to f and $(\phi')^{-1}$,

$$N_f : C \rightarrow C, \quad N_f(u)(r) = f(u(r)),$$

$$N_{(\phi')^{-1}} : C \rightarrow C, \quad N_{(\phi')^{-1}}(u)(r) = (\phi')^{-1}(u(r)).$$

Set $\rho_i > 0$ and denote with $B_{\rho_i} := \{u \in C_\delta \mid \|u\|_\infty \leq \rho_i\}$. We set the following fixed point problem: for any $\zeta_i \in \mathbb{R}$ we want to find $u \in \zeta_i + B_{\rho_i}$ such that

$$u = T_{\zeta_i} \circ K \circ N_{(\phi')^{-1}} \circ S \circ (\lambda N_f(u)). \tag{8}$$

Since $(\phi')^{-1}$ and f are respectively Lipschitz and locally Lipschitz, Banach-Caccioppoli fixed point theorem guarantees the existence of a sufficiently small $\delta > 0$ such that the function $u_i(\lambda) := u(\zeta_i, r) \in \zeta_i + B_{\rho_i}$ is a solution of (8). It is easy to see that u_i is a local solution of the Cauchy problem $(5)_i$.

Let $R_{\zeta_k} > 0$ be such that $[0, R_{\zeta_k})$ is the maximal interval where the function u_k is defined, here $k = 1, 2, \dots, n$. Multiplying $(5)_k$ by u'_k and integrating over $(0, r)$ we obtain the following equality for any $r \in (0, R_k)$:

$$H(u'_k(r)) + (N-1) \int_0^r \frac{[u'_k(s)]^2}{s \sqrt{1 - [u'_k(s)]^2}} = \lambda[F(\zeta_k) - F(u_k(r))], \quad (9)$$

where $H(t) := \frac{1 - \sqrt{1-t^2}}{\sqrt{1-t^2}}$.

For each $k \in \{1, 2, \dots, n\}$, let $I_k = (\alpha_k, \beta_k)$, and take $\zeta_k \in I_k$. By (A2) and (A4), for every $s \in (\beta_{k-1}, \beta_k]$, we have $F(s) \geq F(\alpha_k)$. Thus from (9), we deduce that $H(u'_k(r))$ is bounded as far as $\beta_{k-1} \leq u \leq \beta_k$. Obviously, since $f(u_k(0)) = f(\zeta_k) > 0$, from Eq.(5) $_k$ we deduce that $u''_k(0) < 0$ and this implies that there exists $\sigma > 0$ such that

$$u'_k(r) < 0 \quad \text{and} \quad 0 < u_k(r) < \zeta_k \quad \text{for } 0 < r < \sigma.$$

Set

$$\bar{R}_{\zeta_k} := \begin{cases} \inf\{r \in (0, R_{\zeta_k}) \mid u'_k(r) \geq 0\}, & \text{if } u'_k(r) = 0 \text{ for some } r \in (0, R_{\zeta_k}), \\ +\infty, & \text{otherwise.} \end{cases} \quad (10)$$

From [7, Remark 1.1], it follows that $0 < \sigma \leq \bar{R}_{\zeta_k} \leq +\infty$ and for every $r \in (0, \bar{R}_{\zeta_k})$,

$$\exists \varepsilon > 0 \text{ such that for any } r \in (0, \bar{R}_{\zeta_k}), \quad |u'_k(r)| \leq 1 - \varepsilon. \quad (11)$$

In particular, $\bar{R}_{\zeta_k} = +\infty$ implies $R_{\zeta_k} = +\infty$.

Define the following two classes of intervals

$$I_k^+ := \{\zeta_k \in I_k \mid \exists R'_k \leq R_{\zeta_k} \text{ such that } u_k(r) > 0, u'_k(r) < 0 \text{ for } r \in (0, R'_k), u'_k(R'_k) = 0\},$$

and

$$I_k^- := \{\zeta_k \in I_k \mid \exists R'_k \leq R_{\zeta_k} \text{ such that } u_k(r) > 0, u'_k(r) < 0 \text{ for } r \in (0, R'_k), u_k(R'_k) = 0\}.$$

We will prove that the sets I_k^+ and I_k^- are non-empty, disjoint and open, $k = 1, 2, \dots, n$. Moreover, I_k^+ and I_k^- do not cover I_k .

Lemma 2.1. Assume that $R_{\zeta_k} = +\infty$. For any fixed $\lambda > 0$ and $k \in \{1, 2, \dots, n\}$, $\zeta_k \in (0, \infty)$ be such that $u_k(r) > 0$ for all $r \geq 0$ and $u'_k(r) < 0$ for all $r > 0$. Then the number $l = \lim_{r \rightarrow \infty} u_k(r)$

satisfies

$$f(l) = 0.$$

Furthermore, if f satisfies (A2) and (A5), then $l = 0$.

Proof. Clearly, there exists $l = \lim_{r \rightarrow +\infty} u(r) \geq 0$. By (5)_k and (11), we imply that

$$\lim_{r \rightarrow +\infty} \left(\frac{u'_k(r)}{\sqrt{1 - [u'_k(r)]^2}} \right) = -\lambda f(l). \quad (12)$$

Suppose that $f(l) \neq 0$, say $f(l) > 0$. By simple computations, together with (11) and (12), we deduce that, definitively, $u''_k(r) < -\delta < 0$ for some $\delta > 0$. Of course this is not possible because of (11). Therefore, $f(l) = 0$.

Now, we claim that $l = 0$.

To this end, we only need to prove that for $k = 1$, $l \neq \alpha_1$ and for each $k \in \{2, \dots, n\}$, $l \neq \alpha_i$, $i = 1, 2, \dots, k$, $l \neq \beta_j$, $j = 1, 2, \dots, k-1$. We divide into three steps.

Step 1. We show that for $k = 1$, $l \neq \alpha_1$.

If $N = 2$ and, by contraction, $l = \alpha_1$. Since for any $r > 0$, $\alpha_1 < u_1(r) < \beta_1$, from (7) we deduce that $r\phi'(u'_1(r))$ is decreasing in $[0, +\infty)$ and then, in particular, there exist $R_1 > 0$ and $\delta > 0$ such that for any $r > R_1$, we have $\phi'(u'_1(r)) < -\frac{\delta}{r}$. By (11) we infer that, for some $M_1 > 0$, we have $M_1 u'_1(r) \leq \phi'(u'_1(r))$ and then

$$u'_1(r) \leq -\frac{\delta}{M_1 r} \quad \text{for any } r > R_1.$$

Integrating in (R_1, r) we obtain

$$u_1(r) \leq u_1(R_1) - \frac{\delta}{M_1} \ln \left(\frac{r}{R_1} \right) \rightarrow -\infty \quad \text{as } r \rightarrow +\infty,$$

which contradicts $l = \alpha_1$.

If $N \geq 3$, and suppose on the contrary that $l = \alpha_1$, then computing in (5)₁, we have that the following equality holds in $(0, +\infty)$:

$$\frac{u''_1}{[1 - (u'_1)^2]^{\frac{3}{2}}} = -\frac{N-1}{r} \frac{u'_1}{\sqrt{1 - (u'_1)^2}} - \lambda f(u_1).$$

Taking into account (11), there exists $\delta > 0$ such that $\delta \leq \sqrt{1 - (u'_1)^2} \leq 1$. We deduce that

$$u''_1 = -\frac{N-1}{r} u'_1 [1 - (u'_1)^2] - \lambda f(u_1) [1 - (u'_1)^2]^{\frac{3}{2}} \leq -\frac{N-1}{r} u'_1 - \delta^3 \lambda f(u_1), \quad (13)$$

where we have used the fact that $u'_1 < 0$ and $f(u_1) > 0$. Now we proceed as in [7,8], repeating the arguments for completeness. If we set $v = r^{\frac{N-1}{2}}(u_1 - \alpha_1)$, by (13) we get the following

estimate

$$v'' \leq \left\{ \frac{(N-1)(N-3)}{4r^2} - \delta^3 \lambda \frac{f(u_1)}{u_1 - \alpha_1} \right\} v \quad (14)$$

from which, in view of (A5), we deduce that v'' is definitively negative. Now, since v' is definitively decreasing, certainly there exists $L = \lim_{r \rightarrow +\infty} v'(r) < +\infty$.

However, L cannot be negative, since otherwise $\lim_{r \rightarrow +\infty} v(r) = -\infty$, this is a contradiction. On the other hand, if $L \geq 0$, then we deduce that v is definitively increasing and then there exists $R_1 > 0$ such that for any $r > R_1$, we have $v(r) > v(R_1)$. From (14) we infer that, for some positive constant C , $v''(r) \leq -C < 0$ definitively and this implies $L = \lim_{r \rightarrow +\infty} v'(r) = -\infty$, again a contradiction.

Step 2. we show that for $k = 2$, $l \neq \alpha_1$, α_2 and $l \neq \beta_1$.

By a similar argument as step 1 with $u_2(r)$ instead of $u_1(r)$, and $v_{2,i} = r^{\frac{N-1}{2}}[u_2(r) - \alpha_i]$, $i = 1, 2$ instead of v , we can deduce that $l \neq \alpha_i$, $i = 1, 2$. Notice that when $N = 2$, we prove $l \neq \alpha_1$, by contradiction, suppose $\lim_{r \rightarrow +\infty} u_2(r) = \alpha_1$, this implies that there exists $R_2 > 0$ large enough such that $\alpha_1 \leq u_2(r) \leq \beta_1$ for $r > R_2$, by a same argument as step 1, which deduce a contradiction. So, we only need to show $l \neq \beta_1$.

Suppose on the contrary that $l = \beta_1$. If $N = 2$, then it follows from $\lim_{r \rightarrow +\infty} u_2(r) = \beta_1$ that there exists $\tilde{R}_2 > 0$ large enough such that for any $r > \tilde{R}_2$, $\beta_1 < u_2(r) < \alpha_2$. If $N = 2$, and $l = \beta_1$, from (7) we deduce that $r\phi'(u'(r))$ is increasing in $[\tilde{R}_2, +\infty)$ and then, in particular, there exist $R_2 > \tilde{R}_2$ and $\delta_1 > 0$ such that for any $r > R_2$, we have $\phi'(u'(r)) > \frac{\delta_1}{r}$. By (11) we infer that, for some $M_2 > 0$, we have $M_2 u'(r) \geq \phi'(u'(r))$ and then

$$u'(r) \geq \frac{\delta_1}{M_2 r} \quad \text{for any } r > R_2.$$

Integrating in (R_2, r) we obtain

$$u(r) \geq u(R_2) + \frac{\delta_1}{M_2} \ln\left(\frac{r}{R_2}\right) \rightarrow +\infty \quad \text{as } r \rightarrow +\infty,$$

which contradicts $l = \beta_1$.

If $N \geq 3$, then computing in (5)₂, we have that the following equality holds in $(0, +\infty)$:

$$\frac{u_2''}{[1 - (u_2')^2]^{\frac{3}{2}}} = -\frac{N-1}{r} \frac{u_2'}{\sqrt{1 - (u_2')^2}} - \lambda f(u_2).$$

Taking into account (11), there exists $\delta_2 > 0$ such that $\delta_2 \leq \sqrt{1 - (u_2')^2} \leq 1$. We deduce that

$$u_2'' = -\frac{N-1}{r} u_2' [1 - (u_2')^2] - \lambda f(u_2) [1 - (u_2')^2]^{\frac{3}{2}} \geq -\delta_2^2 \frac{N-1}{r} u_2' - \delta_2^3 \lambda f(u_2), \quad (15)$$

where we have used the fact that $u'_2 < 0$ and $f(u_2) < 0$ on $[\tilde{R}_2, \infty)$.

Let $w(r) = r^{\frac{\delta^2(N-1)}{2}}(\beta_1 - u(r))$. It follows that

$$\begin{aligned} w'(r) &= \frac{\delta^2(N-1)}{2} r^{\frac{\delta^2(N-1)-2}{2}}(\beta_1 - u(r)) - r^{\frac{\delta^2(N-1)}{2}} u'(r), \\ w''(r) &= -r^{\frac{\delta^2(N-1)}{2}} u''(r) - \delta^2(N-1) r^{\frac{\delta^2(N-1)-2}{2}} u'(r) + \frac{\delta^2(N-1)[\delta^2(N-1)-2]}{4r^2} r^{\frac{\delta^2(N-1)}{2}}(\beta_1 - u(r)). \end{aligned}$$

This together with inequality (15) implies that

$$\begin{aligned} w''(r) &\leq r^{\frac{\delta^2(N-1)}{2}} \left[\delta^2 \frac{N-1}{r} u'(r) + \delta^3 \lambda f(u) \right] - \delta^2(N-1) r^{\frac{\delta^2(N-1)-2}{2}} u'(r) \\ &\quad + \frac{\delta^2(N-1)[\delta^2(N-1)-2]}{4r^2} r^{\frac{\delta^2(N-1)}{2}}(\beta_1 - u(r)) \\ &= \delta^2 \frac{N-1}{r} r^{\frac{\delta^2(N-1)}{2}} u'(r) - \delta^2 \frac{N-1}{r} r^{\frac{\delta^2(N-1)}{2}} u'(r) \\ &\quad + \left[\delta^3 \lambda \frac{f(u)}{\beta_1 - u} + \frac{\delta^2(N-1)[\delta^2(N-1)-2]}{4r^2} \right] w(r) \\ &= \left[\delta^3 \lambda \frac{f(u)}{\beta_1 - u} + \frac{\delta^2(N-1)[\delta^2(N-1)-2]}{4r^2} \right] w(r), \end{aligned} \tag{16}$$

from which, in view of (A5), we deduce that w'' is definitively negative. Now, since w' is definitively decreasing, certainly there exists $L = \lim_{r \rightarrow +\infty} w'(r) < +\infty$. However, L cannot be positive, since otherwise $\lim_{r \rightarrow +\infty} w(r) = +\infty$, this is a contradiction. On the other hand, if $L \leq 0$, then we deduce that there exists $R_2 > 0$ such that

$$w'(r) \leq 0, \quad r > R_2,$$

and w is definitively decreasing. Hence, there exist two constants R_* and σ with $R_* > R_2$ and $\sigma > 0$, such that

$$w(r) < 0, \quad w''(r) \leq \sigma^2 w(r), \quad r \in [R_*, +\infty).$$

Set $b_1 := w(R_*)$, $b_2 := w'(R_*)$. Then $b_1 < 0$, $b_2 \leq 0$. Let us consider the initial value problem

$$x''(r) = \sigma^2 x(r), \quad r \in (R_*, \infty), \quad x(R_*) = b_1, \quad x'(R_*) = b_2.$$

Its unique solution can be explicitly given by

$$x(r) = \frac{b_1 - b_2}{2} e^{-\sigma(r-R_*)} + \frac{b_1 + b_2}{2} e^{\sigma(r-R_*)}, \quad r \in (R_*, \infty).$$

Let $z(r) = x(r) - w(r)$. Then

$$z''(r) \geq \sigma^2 z(r), \quad r \in (R_*, \infty), \quad z(R_*) = 0, \quad z'(R_*) = 0.$$

Let

$$M(r) := z''(r) - \sigma^2 z(r), \quad r \in (R_*, \infty).$$

Then

$$z''(r) - \sigma^2 z(r) = M(r), \quad r \in (R_*, \infty), \quad z(R_*) = 0, \quad z'(R_*) = 0,$$

which has a unique solution

$$z(r) = \frac{1}{2\sigma} \int_{R_*}^r [e^{\sigma(r-s)} - e^{-\sigma(r-s)}] M(s) ds, \quad r \in (R_*, \infty).$$

Obviously, $z(r) \geq 0$ for $r \in (R_*, \infty)$, which implies

$$x(r) \geq w(r), \quad r \in (R_*, \infty),$$

i. e.

$$\frac{b_1 - b_2}{2} e^{-\sigma(r-R_*)} + \frac{b_1 + b_2}{2} e^{\sigma(r-R_*)} \geq r^{\frac{\delta^2(N-1)}{2}} (\beta_1 - u(r)), \quad r \in (R_*, \infty).$$

However, this is impossible since

$$\frac{b_1 + b_2}{2} < 0, \quad \beta_1 - \zeta_2 < \beta_1 - u(r) \leq 0, \quad \lim_{r \rightarrow \infty} \frac{e^{\sigma(r-R_*)}}{r^{\frac{\delta^2(N-1)}{2}}} = +\infty.$$

Therefore, $l \neq \beta_1$.

Step 3. We claim that for each $k \in \{3, 4, \dots, n\}$, $l \neq \alpha_i$, $i = 1, 2, \dots, k$ and $l \neq \beta_j$, $j = 1, 2, \dots, k-1$.

Toward this end, we only need to repeat the arguments of step 1 and step 2 with $u_k(r)$ instead of $u_1(r)$, and $v_{k,i} = r^{\frac{N-1}{2}} [u_k(r) - \alpha_i]$, $i = 1, 2, \dots, k$ instead of v , and the proof of $l \neq \beta_1$ with u_k instead of u_2 and $w_{k,j}(r) = r^{\frac{N-1}{2}} [\beta_j - u_k(r)]$, $j = 1, 2, \dots, k-1$ instead of w . \square

Lemma 2.2. For any fixed $\lambda > 0$ and let $k \in \{1, 2, \dots, n\}$, $I_k^+ \neq \emptyset$.

Proof. Let $\zeta_k \in (\alpha_k, \xi_k]$. By (A2) and (A3), $F(\zeta_k) < 0$. Because of (9) and the definition of ξ_k , it is clear to see that $F(u(r)) < F(\zeta_k) < 0$ for any $r \in (0, R_{\zeta_k})$. As a consequence, by the fact $f(\zeta_k) > 0$ in $(\alpha_k, \xi_k]$ we have that there exists $m_k > 0$ such that

$$0 < m_k < u_k(r) < \zeta_k. \quad (17)$$

Suppose on the contrary that $\zeta_k \notin I_k^+$, then $\bar{R}_{\zeta_k} = +\infty$ implies $R_{\zeta_k} = +\infty$. So $u'(r) < 0$ for any $r > 0$, by Lemma 2.1 we get a contradiction with (17). \square

Next, we will prove that I_k^- is not empty, we need some preliminary results.

For each $i \in \{1, 2, \dots, n\}$, consider the problem

$$\begin{aligned} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right] + \lambda f(u) &= 0, & \text{in } B_\rho, \\ u &= 0, & \text{on } \partial B_\rho. \end{aligned} \quad (18)$$

Recall the definition of β_i , we replace f in (18) by

$$f_i(s) = \begin{cases} f(s), & \text{if } s \leq \beta_i, \\ f(\beta_i), & \text{if } s > \beta_i. \end{cases} \quad (19)$$

As in [3, 7], we use a variational approach to (18).

Set $W_\rho := W^{1,\infty}((0, \rho), \mathbb{R})$. It is well known that $W_\rho \hookrightarrow C_\rho$. Define

$$K := \{u \in W_\rho \mid \|u'\|_\infty \leq 1, \quad u(\rho) = 0\}$$

and

$$\Psi(u) := \begin{cases} \int_0^\rho r^{N-1} \left(1 - \sqrt{1 - (u')^2}\right) dr, & \text{if } u \in K, \\ +\infty, & \text{if } u \in W_\rho \setminus K. \end{cases}$$

For any $u \in W_\rho$, we set

$$J_i(\lambda, u) := \Psi(u) - \lambda \int_0^\rho r^{N-1} F_i(u) dr.$$

It is easy to verify that the functional $J_i(\lambda, \cdot)$ is a Szulkin's functional (see [12]) so that, by [12, Proposition 1.1], we have that if $u \in W_\rho$ is a local minimum of $J_i(\lambda, \cdot)$, then it is a Szulkin critical point and for any $v \in K$ it solves the inequality

$$\int_0^\rho r^{N-1} (\phi(v') - \phi(u')) dr - \lambda \int_0^\rho r^{N-1} f_i(u)(v - u) dr \geq 0, \quad (20)$$

where we recall that ϕ is defined in (7). By a similar argument from [7, 13], we obtain the following lemma.

Lemma 2.3. For all $\lambda > 0$, if $v_i(\lambda, \cdot) \in K$ is a local minimum for $J_i(\lambda, \cdot)$, then $v_i(\lambda, |x|)$ is a classical solution of (18) for each $i \in \{1, 2, \dots, n\}$.

Lemma 2.4. For all $\lambda > 0$, $\forall \rho > 0$ and for each $i \in \{1, 2, \dots, n\}$, there exists $v_i(\lambda, \cdot) \in K$ such that $J_i(\lambda, \cdot)$ attains its local minimum at $v_i(\lambda, \cdot)$.

Moreover, $v_i(\lambda, \cdot)$ is a classical nontrivial solution of (18) and satisfies $0 \leq v_i(\lambda, \cdot) \leq \beta_i$.

Proof. As a first step, we show that $J_i(\lambda, \cdot)$ is bounded below and achieves its infimum.

Observe that $\forall v \in K, \|v\|_\infty \leq \rho$. As a consequence, it is easy to see that $J_i(\lambda, \cdot)$ is bounded below. Consider $\{v_{i,k}\}_{k=1}^\infty \in W_\rho$ a minimizing sequence. Of course we can assume $v_{i,k} \in K$ for any $k \geq 1$. By the Ascoli Arzelà theorem, there exists a subsequence, relabeled $\{v_{i,k}\}_{k=1}^\infty$, and a continuous function v_i^* such that

$$v_{i,k} \rightarrow v_i^* \quad \text{uniformly in } [0, \rho]. \quad (21)$$

To prove that v_i^* is in K , we just observe that, for any $x, y \in [0, \rho]$ with $x \neq y$, we have

$$\lim_k \frac{v_{i,k}(x) - v_{i,k}(y)}{x - y} = \frac{v_i^*(x) - v_i^*(y)}{x - y},$$

and then also v_i^* has Lipschitz constant 1. By (21) and [13, Lemma 1], it deduce that $\Psi(v_i^*) \leq \liminf_k \Psi(v_{i,k})$. Then, again by (21), we have

$$J_i(\lambda, v_i^*) \leq c_{i,0},$$

where $c_{i,0} = \inf_{v \in W_\rho} J_i(\lambda, v)$.

Now we claim that if $\rho > 0$ is sufficiently large, then $c_{i,0} < 0$. Consider the following function defined for $\rho > 2\gamma_i$,

$$\omega_\rho(r) = \begin{cases} \gamma_i, & \text{in } [0, \rho - 2\gamma_i], \\ \frac{\rho - r}{2}, & \text{in } [\rho - 2\gamma_i, \rho]. \end{cases}$$

Of course $\omega_\rho \in K$. Moreover

$$\begin{aligned} J_i(\lambda, \omega_\rho) &\leq \frac{1}{2} \int_{\rho - 2\gamma_i}^\rho (2 - \sqrt{3}) s^{N-1} ds - F(\gamma_i) \frac{(\rho - 2\gamma_i)^N}{N} + \frac{1}{N} \max_{0 \leq s \leq \gamma_i} |F(s)| [(\rho)^N - (\rho - 2\gamma_i)^N] \\ &\leq C_1 [\rho^N - (\rho - 2\gamma_i)^N] - \frac{F(\gamma_i)(\rho - 2\gamma_i)^N}{N} \\ &\leq C_2 \rho^{N-1} - C_3 \rho^N < 0 \quad \text{as } \rho > 2\gamma_i \text{ sufficiently large,} \end{aligned}$$

where C_1 , C_2 and C_3 are suitable positive constants. The claim is an obvious consequence of the previous chain of inequalities. This together with Lemma 2.3 yields the conclusion. \square

We will use a similar method in [14, Lemma 2.5] to obtain an important lemma.

Lemma 2.5. If $\lambda > 0$ is sufficiently large, then $\sup\{v_{i+1}(\lambda, r) \mid r \in B_\rho\} > \beta_i$ and consequently, $v_{i+1}(\lambda, \cdot) \neq v_i(\lambda, \cdot)$, $i \in \{1, 2, \dots, n-1\}$.

Proof. To this end, we only need to show that there exists $w \in K$ such that $J_{i+1}(\lambda, w) < J_{i+1}(\lambda, v)$ for all $v \in K$ satisfying $0 \leq v \leq \beta_i$.

First of all, we show that for $\lambda > 0$ is sufficiently large, then $\sup\{v_2(\lambda, r) \mid r \in B_\rho\} > \beta_1$ and subsequently $v_2(\lambda, \cdot) \neq v_1(\lambda, \cdot)$.

Let $\varrho_0 = \inf\{F(\beta_2) - F(v(r)) : r \in \bar{B}_\rho \text{ and } 0 \leq v \leq \beta_1\}$. Then $\varrho_0 > 0$ as f satisfies the condition (A4). If $v \in K$ satisfies $0 \leq v \leq \beta_1$, then

$$\begin{aligned} \int_0^\rho r^{N-1} F_2(v(r)) dr &= \int_0^\rho r^{N-1} F(u(r)) dr \\ &\leq \int_0^\rho r^{N-1} F(\beta_2) dr - \frac{\rho^N}{N} \varrho_0 = F(\beta_2) \frac{\rho^N}{N} - \varrho_0 \frac{\rho^N}{N}. \end{aligned} \tag{22}$$

On the other hand, let $\rho > 2\beta_2$, consider the following function

$$w_\rho(r) = \begin{cases} \beta_2, & \text{in } [0, \rho - 2\beta_2], \\ \frac{\rho - r}{2}, & \text{in } [\rho - 2\beta_2, \rho]. \end{cases}$$

Obviously, $w_\rho \in K$ and

$$\begin{aligned}
\int_0^\rho r^{N-1} F(w_\rho(r)) dr &= \int_0^{\rho-2\beta_2} r^{N-1} F(\beta_2) dr + \int_{\rho-2\beta_2}^\rho r^{N-1} F\left(\frac{\rho-r}{2}\right) dr \\
&= \int_0^\rho r^{N-1} F(\beta_2) dr - \int_{\rho-2\beta_2}^\rho r^{N-1} F(\beta_2) dr + \int_{\rho-2\beta_2}^\rho r^{N-1} F\left(\frac{\rho-r}{2}\right) dr \\
&\geq F(\beta_2) \frac{\rho^N}{N} - 2 \sup_{u \in [0, \beta_2]} |F(u)| \frac{\rho^N - (\rho - 2\beta_2)^N}{N}.
\end{aligned} \tag{23}$$

By (22) and (23) we can choose and fix $\rho > 2\beta_2$ sufficiently large so that

$$\begin{aligned}
&\int_0^\rho r^{N-1} F(w_\rho(r)) dr - \int_0^\rho r^{N-1} F(v(r)) dr \\
&\geq \int_0^\rho r^{N-1} F(w_\rho(r)) dr - \int_0^\rho r^{N-1} F(v(r)) dr \\
&\geq \varrho_0 \frac{\rho^N}{N} - 2 \sup_{v \in [0, \beta_2]} |F(v)| \frac{\rho^N - (\rho - 2\beta_2)^N}{N} \\
&\geq C_4 \rho^N - C_5 \rho^{N-1} > 0, \quad \forall 0 \leq v \leq \beta_1,
\end{aligned}$$

here C_4, C_5 are suitable positive constants. Thus, there exists $\sigma_1 > 0$ such that

$$\int_0^\rho r^{N-1} F(w_\rho(r)) dr - \int_0^\rho r^{N-1} F(v(r)) dr > \sigma_1$$

for all $0 \leq v \leq \beta_1$. Moreover, for such $\rho > 2\beta_2$, it follows that

$$\begin{aligned}
&J_2(\lambda, w_\rho) - J_2(\lambda, v) \\
&= \int_0^\rho r^{N-1} [1 - \sqrt{1 - (w'_\rho(r))^2}] dr - \int_0^\rho r^{N-1} [1 - \sqrt{1 - (v'(r))^2}] dr \\
&\quad - \lambda \int_0^\rho r^{N-1} [F(w_\rho(r)) - F(v(r))] dr \\
&\leq \frac{1}{2} \int_{\rho-2\beta_2}^\rho (2 - \sqrt{3}) r^{N-1} dr - \lambda \int_0^\rho r^{N-1} [F(w_\rho(r)) - F(v(r))] dr \\
&\leq \frac{2 - \sqrt{3}}{2N} [\rho^N - (\rho - 2\beta_2)^N] - \lambda \sigma_1 \\
&\leq 0 \quad \text{for } \lambda \text{ sufficiently large.}
\end{aligned}$$

Hence, for such λ , the local minimum of $J_2(\lambda, \cdot)$ cannot be attained at any $v \in W_\rho$ such that $0 \leq v \leq \beta_1$. Therefore, $\sup\{v_2(\lambda, r) \mid r \in B_\rho\} > \beta_1$ and so $v_2(\lambda, \cdot) \neq v_1(\lambda, \cdot)$.

By the same argument with obvious changes, we can obtain that $\sup\{v_{i+1}(\lambda, r) \mid r \in B_\rho\} > \beta_i$ and $v_{i+1}(\lambda, \cdot) \neq v_i(\lambda, \cdot)$, $i \in \{1, 2, \dots, n-1\}$ for λ sufficiently large. \square

From Lemma 2.3 to Lemma 2.5, it deduce that for any fixed $\rho > 0$ large enough and $k \in \{2, 3, \dots, n\}$, the problem (18) with f_k instead of f has k distinct nontrivial solutions and the k -th solution v_k satisfying $\sup\{v_k(\lambda, r) \mid r \in B_\rho\} > \beta_{k-1}$ with $\lambda > 0$ sufficiently large.

Lemma 2.6 Let $\lambda > 0$ be sufficiently large and $i \in \{1, 2, \dots, n\}$. Then $I_i^- \neq \emptyset$.

Proof From Lemma 2.4, it follows that $J_i(\lambda, \cdot)$ is bounded below and achieves its infimum. Moreover, if $\rho > 0$ is sufficiently large, then $c_{i,0} < 0$.

Now choose $\rho_i > 0$ large enough such that there exists $u_i := v_i(\lambda, \cdot) \in K_i$ satisfying $J_i(u_i) = c_{i,0} < 0$ and $\sup\{u_i(r) \mid r \in B_{\rho_i}\} > \beta_{i-1}$. Set $\tilde{\zeta}_i = u_i(0)$. Then the value $\tilde{\zeta}_i \in (\alpha_i, \beta_i)$. Indeed, by Lemma 2.3 and Lemma 2.5, $u_i(|\cdot|)$ is a classical solution of (18) with ρ_i instead of ρ , and then u_i is a local solution of $(5)_i$ with $\zeta_i = \tilde{\zeta}_i$ and f_i instead of f . If $\tilde{\zeta}_i \leq \alpha_i$, then $\tilde{\zeta}_i \in (\beta_{i-1}, \alpha_i]$ such that $F(\tilde{\zeta}_i) \leq 0$ leads to an obvious contradiction to (9) computed in $r = \rho_i$. On the other hand, $\tilde{\zeta}_i$ can not be greater than β_i , since in this case, by (19), the unique solution of the Cauchy problem $(5)_i$ would be the constant function $u_i(r) = \tilde{\zeta}_i$.

By contradiction, suppose that $\tilde{\zeta}_i \notin I_i^-$. Since we can assume $u_i(r) > 0$ in $[0, \rho_i)$, otherwise we consider the function u_i restricted to the interval $[0, R'_i)$, where $R'_i := \inf\{r > 0 \mid u_i(r) = 0\}$, our contradiction assumption implies that $\bar{R}_{\tilde{\zeta}_i} \in (0, \rho_i)$ (the definition of $\bar{R}_{\tilde{\zeta}_i}$ is given in (10)).

Computing (9) for $r = \bar{R}_{\tilde{\zeta}_i}$ and for $r = \rho_i$, we respectively have

$$(N-1) \int_0^{\bar{R}_{\tilde{\zeta}_i}} \frac{(u'_i(s))^2}{s\sqrt{1-(u'_i(s))^2}} ds = \lambda[F(\tilde{\zeta}_i) - F(u_i(\bar{R}_{\tilde{\zeta}_i}))], \quad (24)$$

$$H(u'_i(\rho_i)) + (N-1) \int_0^{\rho_i} \frac{(u'_i(s))^2}{s\sqrt{1-(u'_i(s))^2}} ds = \lambda F(\tilde{\zeta}_i). \quad (25)$$

Subtracting (24) from (25), we obtain

$$H(u'_i(\rho_i)) + (N-1) \int_{\bar{R}_{\tilde{\zeta}_i}}^{\rho_i} \frac{(u'_i(s))^2}{s\sqrt{1-(u'_i(s))^2}} ds = \lambda F(u_i(\bar{R}_{\tilde{\zeta}_i})),$$

which implies that $F(u_i(\bar{R}_{\tilde{\zeta}_i})) > 0$.

Since $u'_i(r) < 0$ for any $r \in (0, \bar{R}_{\tilde{\zeta}_i})$, we have that $u''_i(\bar{R}_{\tilde{\zeta}_i}) \geq 0$ and then from the equation of $(5)_i$, it follows that $f(u_i(\bar{R}_{\tilde{\zeta}_i})) \leq 0$. Since f is positive in I_i and $0 < u_i(\bar{R}_{\tilde{\zeta}_i}) < \tilde{\zeta}_i < \beta_i$, certainly $u_i(\bar{R}_{\tilde{\zeta}_i}) \in (\beta_{i-1}, \alpha_i]$. From this we deduce that $F(u_i(\bar{R}_{\tilde{\zeta}_i})) < 0$ and then the contradiction is obtained. \square

Lemma 2.7 For any fixed $\lambda > 0$ sufficiently large and let $k \in \{1, 2, \dots, n\}$, I_k^- and I_k^+ are open and disjoint.

Proof By contradiction, suppose $\bar{\zeta}_k \in I_k^+ \cap I_k^-$. Then, since the solution of $(5)_k$ with $\zeta_k = \bar{\zeta}_k$ is such that $u_k(R'_{\zeta_k}) = u'_k(R'_{\zeta_k}) = 0$, by uniqueness theorem, $u = 0$ is the unique solution of the Cauchy problem

$$\left(\frac{u'_k}{\sqrt{1-(u'_k)^2}} \right)' + \frac{N-1}{r} \frac{u'_k}{\sqrt{1-(u'_k)^2}} + \lambda f(u) = 0, \\ u(R'_{\zeta_k}) = 0, \quad u'(R'_{\zeta_k}) = 0.$$

Finally, by continuous dependence on the initial datum, it is easy to see that I_k^+ and I_k^- are open sets. \square

By Lemma 2.2, Lemma 2.6 and Lemma 2.7, for $\lambda > 0$ is sufficiently large, we can take $\zeta_k \in I_k \setminus (I_k^+ \cup I_k^-)$ such that $u_k(r)$ is defined on $[0, \infty)$ and, $u_i \neq u_j$, $i \neq j$. By Lemma 2.1, $\lim_{r \rightarrow +\infty} u_k(r) = 0$. As a consequence, the problem (1) has n distinct positive, decaying radial solution u_k , $k = 1, 2, \dots, n$.

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